A framework for self-organized explosive percolation and synchronization

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Abrupt and explosive phase transitions are one of the most important results in Statistical Physics and in Complex Networks of the last few decades. They are found when the interactions between the elements of the network are coupled with their structural properties. Furthermore, these transitions exhibit drastic and unanticipated consequences that make them an emerging paradigm for modeling real-world systems ranging from social networks or epidemics to nanotubes [1].

The examples more commonly studied are the Explosive Percolation (EP) [2] and the Explosive Synchronization (ES) [3] transition. For both these transitions, the explosive or abrupt nature of the transitions arises from the delay of their critical point. This delay is achieved by introducing correlations between the links of the network and functions of its local structure. Its important to mention that, without this delay of the critical point, both these transitions are smooth and second order transitions.

Our objective in this work is the development and derivation of a rule or set of rules that allow a system to display an explosive behavior in the percolation transition as well as in the synchronization transition.

With this aim in mind, let us consider a system of N non-identical Kuramoto oscillators running on top of a network. The equations of motion are given by

$$\dot{\theta}_i = \omega_i + \lambda \sum_{j=1}^{N} A_{ij} \cdot \sin(\theta_i - \theta_j), \tag{1}$$

where θ_i is the phase of the i–th oscillator, ω_i is its natural frequency and is given by a distribution $g(\omega)$ such that $\langle \omega \rangle = 0$. A_{ij} are the entries of the adjacency matrix $\mathbf{A}$, capturing the interactions between oscillators and λ is the strength of their coupling. The macroscopic behavior of the system is captured by the modulus of the Kuramoto order parameter:

$$r(t) = \frac{1}{N} \left| \sum_{n=1}^{N} e^{i \cdot \theta_n(t)} \right| \quad \text{and} \quad r = \langle r(t) \rangle, \tag{2}$$

which measures the degree of synchronization of the system and is bounded between zero and one.

Now, with a series of assumptions such as: (*i*) the system tends to maximize the synchronization, (*ii*) we have limited information, *i.e.* the process is decentralized, (*iii*) the percolation is adiabatic compared to the synchronization and (*iv*) the system is closed to the synchronization attractor; we are able to derivate a rule for the order in which the links of the network have to be added (removed) that delays the critical point of the percolation and the synchronization transition. This rule is adding (removing) the link (p, q) that maximizes the magnitude

$$\Delta r_{p,q} = \frac{\pm 1}{\lambda^2 N} \left(\frac{\omega_p}{k_p} - \frac{\omega_q}{k_q} \right) \left(\frac{\omega_p}{k_p^2} - \frac{\omega_q}{k_q^2} \right). \tag{3}$$

As we observe in Fig. 1 this rule effectively delays the critical points of both transition, allowing them to display hysteresis.

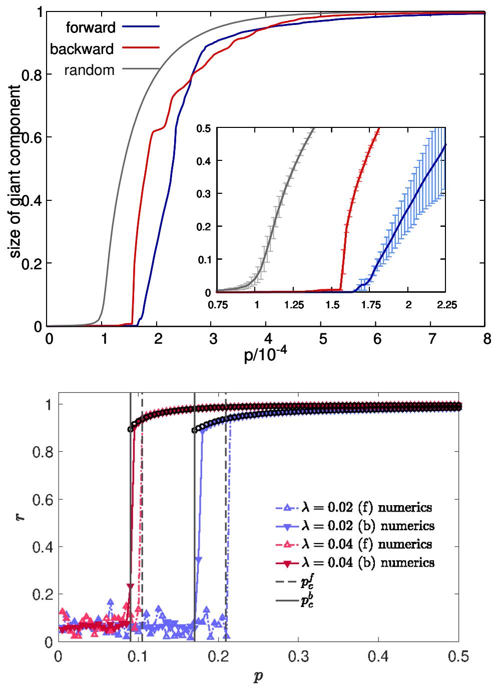

Fig. 1. Percolation and synchronization transitions for a uniform distribution of natural frequencies, $g(\omega)$. Top: Abrupt percolation transition of our system. It can be seen how the critical points of both the forward and backward process are delayed, compared to the random process. Bottom: Explosive synchronization transition of our system. It can be appreciated how the transition, well known to be of second order without any rule, now displays a discontinuity in which the parameter r jumps from $r \approx 0$ to $r \approx 1$ for the forward process and vice versa for the backward process. Black lines correspond to the theoritical predictions for the synchronization thresholds.

[1] R. M. D'Souza, J. Gmez-Gardees, . Arenas and J. Nagler *Advances in Physics* **68**, 3, 123 - 223 (2019).

[2] R. M. D'Souza and J. Nagler, *Nature Physics* **11**, 531 - 538 (2015).

[3] J. Gmez-Gardees, S. Gmez, A. Arenas and Y. Moreno *Physical Review Letters* **106**, 128701 (2011).