Pareto-optimal trade-off: gambling in horse races and growing bacteria

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Developed in 1956 by Bell Labs scientist John Kelly, *Kelly's criterion* applied the newly created field of information theory to gambling and investment [1]. Largely popularized in books [2], this criterion allows a gambler (or investment fund) to fix what proportion of bankroll should be risked on a given bet. It essentially exploits side information to maximize the expected geometric growth rate of a capital. Although ensuring optimal growth rate, Kelly's criterion turns out to be risky, giving rise to large fluctuations in the growth rate.

In many situations, one may consider a possible trade-off between the expected growth and the risk involved, seeking for example to minimize the variance while maximizing the gain, and defining and efficient border in the region of possible strategies. A related idea was introduced in finance by Markowitz [3] under the name of mean-variance analysis as a way to mitigate risk, which in Gaussian models is described by the variance of assets or volatility.

We have studied such efficient border or Pareto front in the original Kelly's horse race problem, finding a trade-off between the average growth rate and its fluctuations [4]. Figure 1 represents the computed fronts for a specific game with 3 horses. We have also found an uncertainty relation between average growth and its standard deviation, resembling those found in Stochastic Thermodynamics [5, 6]. This relation in horse races constrains gambling strategies and imposes a minimal level of risk associated to a certain growth rate, a *no risk no gain* type of bound.

A related idea is found in growing populations, for instance, of monoclonal bacterial colonies subject to fluctuating environments and with stochastic phenotypic switching. Phenotypic switching is usually understood as a bet-hedging strategy to protect the colony against environmental fluctuations. We have analyzed a simple model of two randomly switching phenotypes subjected to two stochastically switching environments, and found similar trade-off curves. Building on these Pareto fronts, our simulations of the dynamics suggest a close connection between the long term variance of the growth rate and the extinction probability, indicating that it may be beneficial for a population to accept a reduction of its short-term reproductive success in exchange for longer-term risk reduction [7].

Bet-hedging is an important topic in biology, associated to a number of phenomena such as species polymorphism, antibiotics resistance of bacteria or the resistance of cancer cells to anti-cancer drugs, and more generally to the phenomenon of cell variability and adaptation by the immune system.

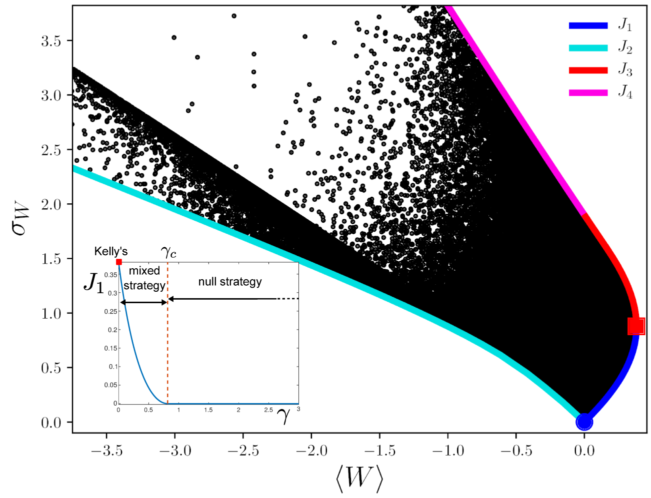

Fig. 1. Pareto borders for 3 horses obtained by simulated annealing built with different utility functions J_1, J_2, J_3 and J_4 (colored solid lines), together with a cloud of points generated by randomly choosing bets satisfying all relevant constraints. Inset: $J_1 = \langle W \rangle - \gamma\sigma_W$ versus γ along the trade-off branch (*i.e.* on the dark blue border) showing the transition from a mixed strategy to a strategy of variance minimization. $\langle W \rangle$ and σ_W stand for the average capital growth rate and its variance, respectively.

[1] J. L. Kelly. A New Interpretation of Information Rate. *Bell System Technical Journal*, 35(4):917–926, 1956.

[2] William Poundstone. *Fortune's Formula: The Untold Story of the Scientific Betting System that Beat the Casinos and Wall Street.* Hill & Wang, 2005.

[3] Harry Markowitz. Portfolio Selection. *The Journal of Finance*, 7(1):77–91, 1952.

[4] L. Dinis, J. Unterberger and D. Lacoste Phase transitions in optimal betting strategies. *Europhys. Lett.* **131**, *60005 (2020)*

[5] Andre C. Barato and Udo Seifert. Thermodynamic Uncertainty Relation for Biomolecular Processes. *Physical Review Letters*, 114(15):158101, April 2015.

[6] Todd R. Gingrich, Jordan M. Horowitz, Nikolay Perunov, and Jeremy L. England. Dissipation Bounds All Steady-State Current Fluctuations. *Physical Review Letters*, 116(12):120601, March 2016.

[7] L. Dinis, J. Unterberger and D. Lacoste. Pareto-optimal trade-off for phenotypic switching of populations in a stochastic environment. *Submitted*