A geometry-induced topological phase transition in random graphs

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Recently, network geometry [1] has become a hot research field in network science. Latent geometric spaces underlying real complex networks provides the simplest explanation to many of their observed topological properties, including degree distribution, smallworldness, clustering, community structure, etc. The key topological property within the geometric framework is clustering—the tendency of the network to form cycles of length three—due to the triangle inequality in the latent geometry. Interestingly, in geometric models clustering undergoes a phase transition between a geometric phase with finite clustering coefficient in the thermodynamic limit and a non-geometric phase where the clustering coefficient is zero.

Despite the fact that the transition was known from our previous works [2], its nature was completely unclear. In this work [3], we analyze this transition in detail and show that it has a quite peculiar behavior. Upon mapping the network ensemble to a system of noninteracting fermions (corresponding to the links in the network) at temperature β^{-1}, we show analytically and confirm numerically that

1. there is no symmetry breaking at the critical point β_c. In fact, the transition is a topological one between two different orderings of chordless cycles (which can be regarded as topological defects) in the network. It is then similar to other topological phase transitions like the BKT transition [4, 5].

2. However, unlike in the BKT transition, both the free energy and entropy of the system diverge at the critical point in the thermodynamic limit. This is a very exotic behavior as in standard systems entropy only diverges at infinite temperature.

3. The scaling behavior of clustering at the transition is anomalous. Right at the critical point clustering decays logarithmically with the system size, and it decays as a power of the system size below the critical point. This is at odds with standard continuous phase transitions, where one observes power law decay at the critical point and faster decay below the critical point.

4. This scaling suggests that the effective size of the system is not given by the number of nodes N but by its logarithm $\ln N$. We then propose a finite size scaling ansatz with $\ln N$ instead of N that is confirmed by both the direct numerical integration as numerical simulation of the problem (Fig. 1).

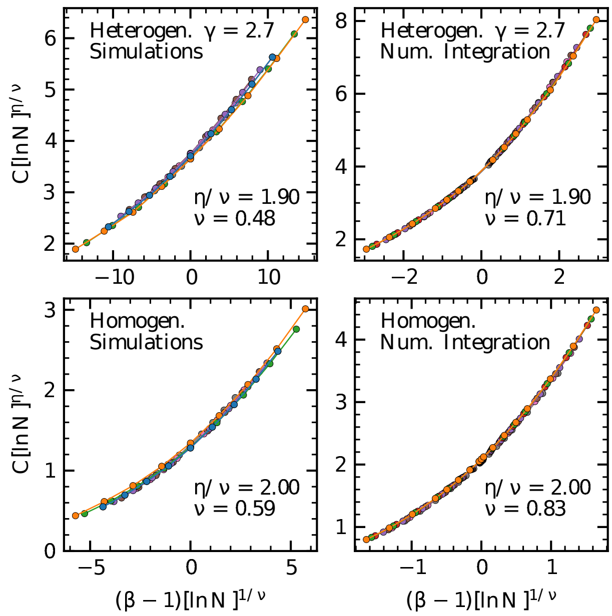

Fig. 1. Data collapse based on the finite size scaling ansatz $C(\beta, N) = [\ln N]^{-\frac{\eta}{\nu}} f((\beta-\beta_c)[\ln N]^{\frac{1}{\nu}})$ for heterogeneous networks with $\gamma = 2.7$ (top row) and homogeneous networks (bottom row). Left column correspond to numerical simulations with sizes in the range $N \in (5 \times 10^2, 10^5)$, whereas the right column is obtained from numerical integration in the range $N \in (5 \times 10^5, 10^8)$.

We also show that real networks with temperatures around and below the critical point are widespread, and therefore justify the practical importance of these findings.

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