Synchronization meets non-pairwise interactions: the enlarged Kuramoto model

Iván León[1], and Diego Pazó[1]
[1]Instituto de Física de Cantabria (IFCA), Universidad de Cantabria-CSIC, 39005 Santander, Spain

The transition from incoherence to collective synchronization is a pivotal phenomenon in a wide variety of systems, from physical to biological. Seeking to understanding the synchronization transition, in 1975 Kuramoto derived his famous model [1]. Applying a perturbative technique (phase reduction) to an ensemble of globally coupled Stuart-Landau oscillators, he derived the first analytically tractable model able to describe the transition to collective synchrony. Despite its great success, the Kuramoto model is not the end of the story. The complex dynamics present in the ensemble of Stuart-Landau oscillators, Fig. 1a), is not captured by the Kuramoto model, Fig. 1b), pointing to the existence of a more complex transition to collective synchronization [2, 3].

In this work we extend the Kuramoto model to shed light on the rich dynamics of the ensemble of Stuart-Landau oscillators. Emulating Kuramoto's idea, we apply phase reduction to second order, obtaining the "enlarged Kuramoto model". The inclusion of new correction terms with non-pairwise interactions give rise to qualitative (in addition to quantitative) differences, Fig. 1 c).

Although the enlarged Kuramoto model is simpler than the ensemble of Stuart-Landau oscillators, it is still difficult to determine the different regimes due to finite size fluctuations. In order to characterize the series of bifurcation occurring in the system, we develop a numerical method that allows an efficient simulation of ensembles of phase oscillators in the thermodynamic limit. This method is based on the decomposition in Fourier-Hermite modes of the oscillator density. Truncating the number of modes, we obtain a finite dimensional system used to simulate the thermodynamic limit of the enlarged Kuramoto model.

The use of Fourier-Hermite modes allows us to analyze the rich dynamics of the model, confirming the existence of a secondary instability and collective chaos in the transition to synchrony. The modes decomposition is applicable to any ensemble of heterogeneous phase oscillators, facilitating and promoting studies of more complicated phase models, such as the ones with higher-order interactions.

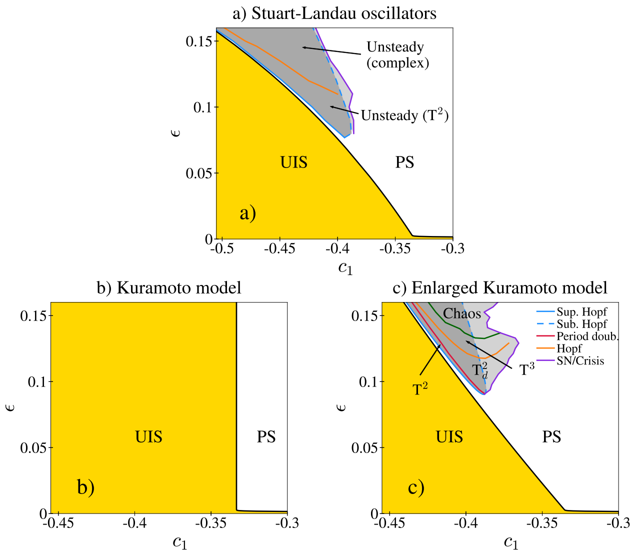

Fig. 1. Phase diagram of a) the ensemble of Stuart-Landau oscillators, b) the Kuramoto model, c) the enlarged Kuramoto model. In yellow and white regions Uniform Incoherence State and Partial Synchrony are stable respectively. The shaded gray region represents the parameters where complex dynamics are stable. Color lines are the transition between different regimes. The rich phase diagram of panel a) is only captured by the enlarged Kuramoto model depicted in panel c).

[1] Y. Kuramoto, *Self-entrainment of a population of coupled nonlinear oscillators*, (International Symposium on Mathematical Problems in Theoretical Physics, Lecture Notes in Physics, Vol. 39, pp. 420-422, edited by Huzihiro Araki, Springer, Berlin, 1975)

[2] I. León and D. Pazó, *Phase reduction beyond the first order: The case of the mean-field complex Ginzburg-Landau equation* Phys. Rev. E **100**, 012211 (2019)

[3] I. León and D. Pazó, *Enlarged Kuramoto Model: Secondary Instability and Transition to Collective Chaos*, arXiv:2112.00176 (2021).