Kuramoto model for populations of quadratic integrate-and-fire neurons with chemical and electrical coupling

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The Kuramoto model (KM) is a minimal mathematical model for investigating the emergence of collective oscillations in populations of heterogeneous, self-sustained oscillators. Over time, it has increasingly been used to describe neuronal oscillations. Yet, the KM was not originally intended to describe any specific natural system, and it is unclear how the parameters of the KM relate to bio-physically meaningful parameters of spiking neuron models.

Here, we uncover a mathematical relation between the Quadratic Integrate-and-Fire (QIF) neuron model and a well-known variant of the KM [1]. By using a perturbation method and averaging, we reduce the population dynamics of weakly heterogeneous QIF neurons that are weakly coupled both with chemical and electrical synapses. The resulting KM readily allows for a comprehensive analysis of the collective dynamics of QIF neurons. We thus provide a rigorous footing for the use of the KM in modeling studies in computational neuroscience. We also show that the ratio of chemical to electrical coupling manifests a critical parameter determining the synchronization properties of the network.

The dynamics of N QIF neurons interacting all-to-all via both electrical and chemical synapses are given by

$$\tau \dot{V}_i = V_i^2 + \eta_i + \varepsilon \left[J\tau r(t) + \tfrac{g}{N} \textstyle\sum_{j=1}^{N} (V_j - V_i) \right], \quad (1)$$

together with the resetting rule: when the membrane potential V_i of neuron $i = 1, \dots, N$ reaches the peak value V_p, neuron i emits a spike and its voltage is reset to V_r. With $V_p = -V_r$ and $V_p \to \infty$, the QIF model is equivalent to the "theta-neuron". We consider nearly identical external currents $\eta_i = \bar{\eta} + \epsilon\chi_i > 0$ so that, in the absence of synaptic inputs ($\varepsilon = 0$), QIF neurons are self-sustained oscillators.

Inhibitory chemical coupling (of strength $-\varepsilon J > 0$) is mediated via the population firing rate $r(t)$. Electrical synapses (of strength $\varepsilon g > 0$) diffusively couple all the neurons with each other, and have previously been shown to favor synchrony in populations of QIF neurons when g is sufficiently strong [2]. Here, however, we consider only weak synapses, $\varepsilon \ll 1$, which allows us to derive the KM for populations of QIF neurons of the form:

$$\dot{\theta}_i = \omega_i + \tfrac{K}{N} \textstyle\sum_{j=1}^{N} \left[\sin(\theta_j - \theta_i - \alpha) + \sin\alpha \right] \quad (2)$$

with natural frequencies $\omega_i = \left(2\sqrt{\bar{\eta}} + \epsilon\chi_i/\sqrt{\bar{\eta}}\right)/\tau$, coupling constant $K = \varepsilon\sqrt{(J/\pi)^2 + g^2}/\tau$, and phase lag parameter $\alpha = \arctan(J/(\pi g))$. The coupling parameters K and α satisfy a simple geometric relation with the coupling parameters of the QIF model Eq. (1), illustrated in Fig. 1.

Using well-known results for the KM [3], we can infer from this geometric relation how chemical and electrical synapses contribute to synchronization: electrical coupling is indispensable for synchrony; in interplay with inhibitory synapses, we find that synchronization depends on both α and the distribution of natural frequencies. For Lorentzian distributed natural frequencies, we can invoke the Ott-Antonsen ansatz and reduce the N-dimensional KM Eq. (2) to a two-dimensional mean field model in the thermodynamic limit $N \to \infty$, which conveniently allows for analyzing the collective dynamics of QIF neurons with both chemical and electrical synapses. To further illustrate the appropriateness of the KM, we investigate the presence of chimera states in two populations of QIF neurons, see Fig. 2, and present a relation between chimera states in spiking neuron networks with those originally uncovered in the KM.

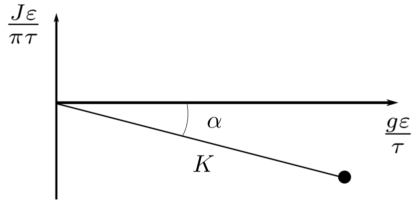

Fig. 1. Geometric relation between the coupling parameters of the QIF model (1) and the Kuramoto model (2).

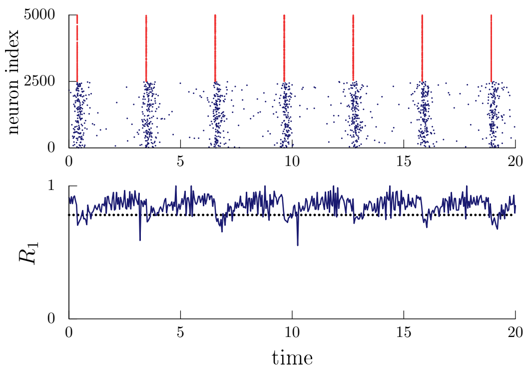

Fig. 2. Chimera state in two-population network of QIF neurons predicted by the Kuramoto model (top). The numerically obtained order parameter R_1 agrees well with theory (bottom).

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