Mastering control of overdamped dynamics: from fast forward/backward to arbitrary connections

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The design and implementation of shortcut protocols able to drive physical systems from an initial state to a target state in a finite time represents a crucial problem in many fields of physics. In the last years, these protocols have been thoroughly studied and applied in quantum systems, commonly under the umbrella of the term *shortcuts to adiabaticity* [1]. More recently, some of these ideas have been adapted and/or exported to different frameworks, from isolated classical systems to statistical mechanics [2]. Still, the picture of a neat correspondence between quantum and classical shortcuts is not complete.

Herein, we present a statistical fast-forward procedure—in the spirit of its quantum counterpart [3]—for one-dimensional overdamped dynamics [4], e.g., a Brownian particle in 1D submitted to a controlled external potential $U(x,t)$. We start from the evolution equation of the probability density function $\rho(x,t)$ describing our system, which is governed by the Smoluchowski equation,

$$\gamma\partial_t\rho(x,t) = \partial_x\left[\rho(x,t)\partial_x U(x,t)\right] + \beta^{-1}\partial_x^2\rho(x,t), \quad (1)$$

where γ stands for the friction coefficient, and $\beta = (k_B T)^{-1}$, with T being the temperature of the thermal bath. From an inverse-engineered perspective, given a prescription $\rho(x,t)$, one can solve Eq. (1) for $U(x,t)$. Nonetheless, in general, the external potential obtained by this method is neither in closed-form nor suitable for applications.

We start by deriving the external potential needed to time-manipulate a certain reference process. Specifically, let us consider a solution $\rho_{\mathrm{r}}(x,t)$ of Eq. (1) with a reference potential $U_{\mathrm{r}}(x,t)$. Therefore, the prescription we would like to impose is

$$\rho(x,t) = \rho_{\mathrm{r}}(x,\Lambda(t)). \quad (2)$$

Note that $\Lambda(t)$ represents a time map between the reference and the manipulated processes. The new and the reference process are the same *film* but played at different *frame rates*, see Fig. 1. Thus, finding the driving potential that enforces the dynamical prescription given by Eq. (2) is providing us with the capability to accelerate, slow down, or even reverse time of a reference process.

Our procedure to accelerate a reference evolution can be used to build arbitrary connections between states. It suffices to use a welding protocol, which matches a fast-forward evolution and a fast-backward (i.e., time reversed) evolution to a common intermediate state. As we will show during the presentation, choosing wisely this intermediate state allows us to work out closed-form expressions for both the driving potential and the evolution itself.

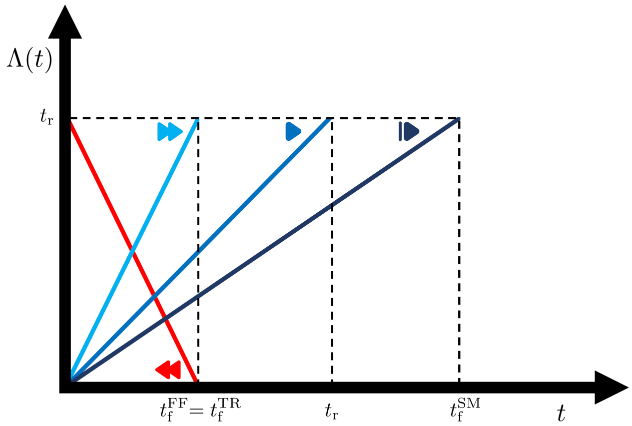

Fig. 1. Sketch of different linear instances of the time manipulation function $\Lambda(t)$. The absolute value of the slope of such function gives the acceleration ($|\dot\Lambda(t)| > 1$) or deceleration ($|\dot\Lambda(t)| < 1$) rate with respect to the reference time evolution. On the one hand, three forward processes has been plotted. Namely, a fast forward (cyan), an identity (blue), and slow motion (purple) transformations with final duration equal to t^{FF}, t_{r}, and $t_{\mathrm{f}}^{\mathrm{SM}}$ respectively. On the other hand, a fast time reversal (red) transformation has been displayed with the same acceleration factor than that of the fast forward example, $t_{\mathrm{f}}^{\mathrm{TR}} = t_{\mathrm{f}}^{\mathrm{FF}}$. Note that, in general, $\Lambda(t)$ does not need to be linear.

[1] D. Guéry-Odelin, A Ruschhaupt, A. Kiely, E. Torrontegui, S. Martínez-Garaot, and J. G. Muga, *Shortcuts to adiabaticity: Concepts, methods, and applications*, Rev. Mod. Phys. **91**, 045001 (2019).

[2] A. Patra and C. Jarzynski, *Shortcuts to adiabaticity using flow fields*, New J. Phys. **19**, 125009 (2017).

[3] S. Masuda and K. Nakamura, *Fast-forward problem in quantum mechanics*, Phys. Rev. A **78**, 062108 (2008).

[4] C. A. Plata, A. Prados, E. Trizac, and D. Guéry-Odelin, *Taming the Time Evolution in Overdamped Systems: Shortcuts Elaborated from Fast-Forward and Time-Reversed Protocols*, Phys. Rev. Lett. **127**, 190605 (2021).