Effect of memory in a dynamic quantum Cournot duopoly game with heterogeneous players

Juan Grau-Climent[1], Luis Garcia-Perez[1], Ramon Alonso-Sanz[1], and Juan Carlos Losada[1]
[1]Complex Systems Group, Universidad Politécnica de Madrid, C. Universitaria, 28040 Madrid, Spain.

The study of Game Theory is widely developed currently and has many applications in several fields such as economics, psychology, biology, etc. The use of entangled modelling is a demonstrated way to improve the results comparing with the classic games due to the entanglement between the players, based on quantum theory. In this kind of models, also called quantum games, it is often studied how the degree of entanglement affects the equilibrium strategies of both players [1]. An interesting case is when the players are heterogeneous, so one of them follows naively the previous steps of the other rational player, since the dynamics of both players can be chaotic and not reach stable equilibria [2].

In the classic Cournots Duopoly with linear inverse demand, the profits of the two players given by:

$$u_i(t) = q_i(t)\left(p(t) - c_i\right), \tag{1}$$

where $q_i(t), i = 1, 2$ represent the output of ith player in the period t, $p(t)$ the prevailing price and c_i the marginal cost of the i player.

Now, we consider the quantization of the classical Cournot's Duopoly, according to Li-Du-Massar quantum scheme [3]. This quantum version has the following steps. First, the game starts from *initial state* $|00\rangle$. This state undergoes a *unitary operation* $\hat{J}(\gamma) = e^{-\gamma\left(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1\hat{a}_2\right)}$, where $a_i^\dagger(\hat{a}_i)$ represents the creation (annihilation) operator of player i and $\gamma \geq 0$ is known as the *squeezing parameter*. Next, the two players execute their *strategic movements* via unitary operation $\hat{D}_i(x_i) = e^{x_i\left(\hat{a}_i^\dagger - \hat{a}_i\right)/\sqrt{2}}, i = 1, 2.$. Finally, the two player' states are measured after *a disentanglement operation* $\hat{J}(\gamma)^\dagger$. Thus, the *final state* is carried out by $|\psi_f\rangle = \hat{J}(\gamma)^\dagger \left(\hat{D}_1(x_1) \otimes \hat{D}_2(x_2)\right) \hat{J}(\gamma)|00\rangle$. The quantity q_i is obtained by acting on the state $|\psi_f\rangle$:

$$\begin{aligned} q_1(t) &= \langle\psi_f|\hat{D}_1(x_1)|\psi_f\rangle = x_1(t)\cosh\gamma + x_2(t)\sinh\gamma, \\ q_2(t) &= \langle\psi_f|\hat{D}_2(x_2)|\psi_f\rangle = x_2(t)\cosh\gamma + x_1(t)\sinh\gamma, \end{aligned} \tag{2}$$

where x_1 and x_2 represent the strategies used by the two firms in the quantum game. When the degree of quantum entanglement is zero, i.e. $\gamma = 0$, then the quantum game turns into the original classical form.

In this study we consider two players with different expectations (heterogeneous expectations), where the first one is a boundedly rational player and the other one is a naive player.The strategy of first player, with bounded rationality, depend on the gradient of their marginal profit:

$$x_1(t+1) = x_1(t) + \alpha x_1(t)\frac{\partial u_1}{\partial x_1(t)}, \tag{3}$$

where α is a parameter which represents the speed of adjustment. The second player, the naive player, expects equal production in each period. Hence, this player will choose the level of output which maximizes the expected profit, i.e:

$$x_2(t+1) = x_2^*(t). \tag{4}$$

where $x_2^*(t)$ satisfies

$$\left|\frac{\partial u_2}{\partial x_2(t)}\right|_{x_2^*(t)} = 0. \tag{5}$$

The dynamic system obtained has the following structure:

$$\begin{aligned} x_1(t+1) &= f(x_1(t), x_2(t), \alpha, \gamma), \\ x_2(t+1) &= g(x_1(t), \alpha, \gamma) \end{aligned} \tag{6}$$

In this work we propose an evolution of quantum games with heterogeneous players that includes memory. In previous works we have explored the effect of embedded memory in chaotic dinamical systems [4]. In this case, the strategy of each player takes into account the strategies of previous time steps in a weighted way. To do this, we replace, in the f and g funtions of ecs. 6, the values x_i given by the following weighted function

$$\mu_i(t) = \frac{x_i(t) + \sum_{\tau=0}^{t-1}\beta^{t-\tau}x_i(\tau)}{1 + \sum_{\tau=0}^{t-1}\beta^{t-\tau}} \tag{7}$$

where the choice of the memory factor β simulates the long-term or remnant memory effect: the limit case $\beta = 1$ corresponds to a memory with equally weight records (*full* memory), whereas $\beta \ll 1$ intensifies the contribution of the most recent states (short-term working memory). The choice $\beta = 0$ leads to the non-memory model.

We have analytically proven and numerically verified that including memory in the dynamics, the stability of the system significantly improves without variation in the fixed points comparing with the memoryless model. Furthermore, even chaotic dynamics disappears in the case of maximum memory.

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