Farey graphs: a real number exotic representation

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This research aims to study some properties of real numbers through the *Farey graphs*, a graph set built recursively using a concatenation operator. The graph structure, especially the connectivity distribution, allows analysing the inherent classification into family of real numbers.

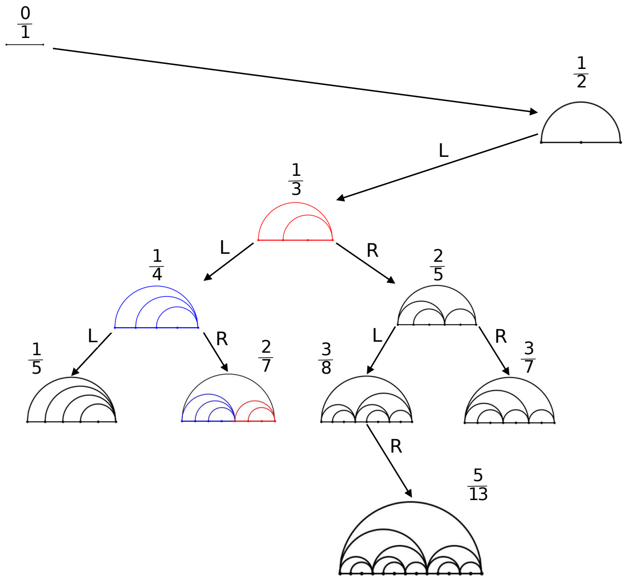

Fig. 1. The first six layers of Farey Graph Tree in $[0, 1/2]$. The rational associated is $P(2)$,the proportion of nodes with degree $k = 2$.

The Farey graph set is constructed using a initial graph (two nodes joined by a link) and an inner operation (see Fig. 2). We prove that there exists a one-to-one correspondance between the Farey graphs and Farey sequences, that are defined as $\mathcal{F}_n = \left\{ \frac{p}{q} \in [0,1] : 0 \leq p \leq q \leq n, \ (p,q) = 1 \right\}$ [1]. Passing to the limit, each real number in $[0, 1]$ can be associated with a Farey graph.

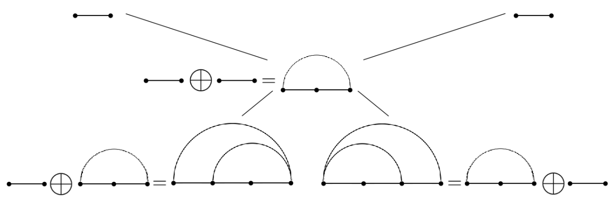

Fig. 2. An illustration of the concatenation operation, that consists on merging two extreme nodes and join the first and last nodes of the new graph with a new link

We are interested in a particular graph entropy over the degree distribution $P(k)$ [2, 3]. We compute this entropy $S(x)$ for all graphs with at least 1000 nodes (see Fig. 3). We show that $S(x)$ is a fractal function where the maximum entropy is associated with the reciprocal of Golden number. The local maximums entropy are the noble numbers, a subfamily of quadratic irrationals having a continued fraction with an infinite sequence of 1. Moreover, the local minimums entropy are the rational numbers.

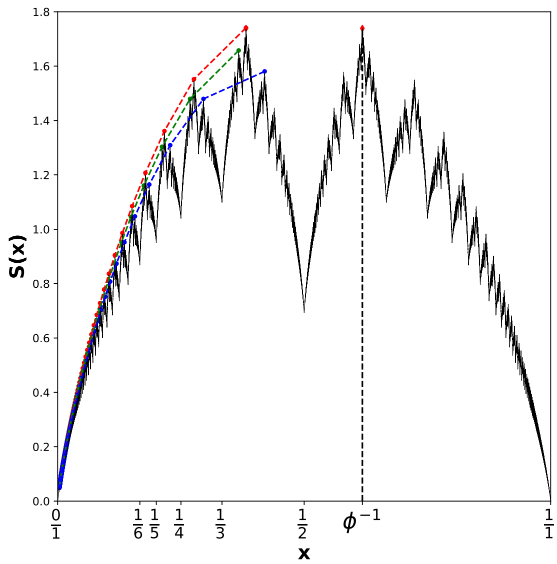

Fig. 3. Number entropy function $S(x)$ computed for all Farey graphs G_x with $x \in \mathcal{F}_{1000}$. In this figure, red dots and dashed lines are the local maxima $\mathcal{C}_1(n) = \frac{1}{n+\phi^{-1}}$ for $n \geq 2$. The global maxima are $\mathcal{C}_1(2) = 1 - \phi^{-1}$ and the reflected ϕ^{-1} (represented as a single red dot). The green and blue dots and dashed lines represent other families of noble numbers.

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