Biased Diffusion-Advection on undirected networks

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Diffusion and advection processes are ubiquitous in natural phenomena. While advection processes show off whenever there is a privileged direction or external force affecting the process, diffusion appears when the system expands in all directions. This two effects are somehow opposite, fighting between following an specific direction and considering all of them equal. Thus, it makes interesting to study how a system evolves when both of them are present.

In the case of networks, diffusion processes are usually modeled by the Laplacian operator, a classical operator widely studied. On the other hand, advection has only been defined in directed networks, as flow orientation is naturally given by the directed edges of the graph. But, how can we generate a flow which is directly related to the graph core properties when the edges are undirected?

The answer comes together with the biased Laplacians. These operators are an extension of the classical diffusion operator and have been studied in the recent years [2]. Its main characteristic is that they their diffusion shows preference or rejects higher degree nodes, depending on whether we are referring to the hubs-attracting or the hubs-rejecting Laplacians. This difference on the diffusion probability generates an underlying bi-directed weighted graph (see fig 1) and this new configuration makes possible to define a vector field on the network, defining a new advection process.

Combining both the classical Laplacian and the new advection operator, we can model this competition between diffusion and advection in any undirected networks. In [1], we defined an Advection-Diffusion operator which depends upon two parameters, γ_{dif} and γ_{adv}, measuring how diffusive and advective is a given system.

We were able to prove that this operators have a stable state and that its convergence is faster than both the classical and the hubs-biased Laplacians alone. Moreover, for a node v_i with degree k_i, we could find an analytic expression for the node value in the final state:

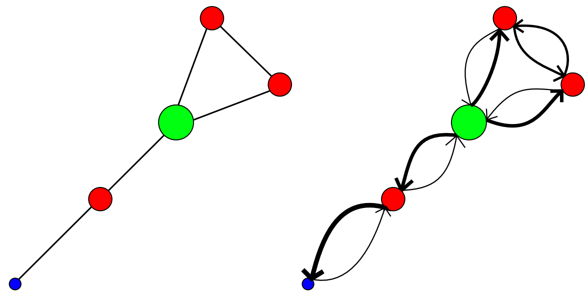

Fig. 1. Original graph (left) compared to the underlying bidirected graph induced by the Hubs-repelling Laplacian (right).

$$v_i = \sum_{T_i \in \mathcal{T}(G)} \prod_{(j,l) \in T_i} \left(\gamma_{dif} + \gamma_{adv} \left(\frac{k_l}{k_j} \right)^{\alpha} \right), \qquad (1)$$

where $\mathcal{T}(G)$ is the set of all spanning trees of the graph G and T_i are each of this spanning trees rooted from node v_i.

We used this new operator to analyze the foraging of *L. catta* in a patched landscape in Southern Madagascar, so we could extract how much diffusion and how much advection there is in this situation.

[1] Miranda, M., & Estrada, E. (2021). *Degree-biased advection-diffusion on undirected graphs/networks*. Preprint: https://hal.archives-ouvertes.fr/hal-03469355

[2] Estrada, E., & Mugnolo, D. (2022). *Hubs-biased resistance distances on graphs and networks*. Journal of Mathematical Analysis and Applications, 507(1), 125728.